izing the $\beta_{i\lambda}$ we find the $\mathbf{b}_{i\lambda}$. Upon execution of these calculations cases are certainly encountered when the scalars $C_{\lambda\lambda}$ in equalities of the type (3.4) turn out to be zero. This means that $\beta_{i\lambda}$ is a linear combination of the $\mathbf{b}_{i\mu}$ $(\lambda > \mu)$. In such cases the vectors $\mathbf{b}_{i\lambda}$ should be omitted and the next vectors $\mathbf{b}_{i,\lambda+1}$ should be considered.

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Translated by M. D. F.

ON THE STABILITY OF ROTATIONAL MOTION OF A VARIABLE COMPOSITION BODY WITH A GYROSCOPE IN A NEWTONIAN FORCE FIELD

PMM Vol. 34, №3, 1970, pp. 564-566
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(Received May 21, 1969)

Sufficient conditions are presented for the stability of rotational motion of a variable-mass body in a central Newtonian force field. The equations of body motion around a fixed point are written under assumptions made by M. Sh. Aminov.

The Chetaev method, as well as the V. V. Rumiantsev theorem on the stability of motion relative to part of the variables, are used in investigating the stability of rotational motions of a solid in the Lagrange case.

Let us consider a symmetric body $(A = B)$ of variable mass on whose axis of symmetry a gyroscope with kinetic moment l_0 is placed and there is the center of mass of the body at a distance $Z_c(t)$ from a fixed point O.

If the body is in a central Newtonian force field, the Euler-Poisson equations, under the assumptions considered in [1-3], have the form

$$p^{\cdot} = (1-\delta)\,qr - \nu q + {}^1/_2 a\gamma_2 - \mu(1-\delta)\gamma_2\gamma_3 \qquad (\nu = l_0 / A)$$

$$q^{\cdot} = (\delta - 1)\,pr + \nu p - {}^1/_2 a\gamma_1 + \mu(1-\delta)\gamma_1\gamma_3, \qquad r^{\cdot} = 0 \qquad (\delta = C / A) \qquad (0.1)$$

$$\gamma_1^{\cdot} = r\gamma_2 - q\gamma_3, \qquad \gamma_2^{\cdot} = p\gamma_3 - r\gamma_1, \qquad \gamma_3^{\cdot} = q\gamma_1 - p\gamma_2 \qquad (a = 2MgZ_c / A)$$

Here ν, δ, a are some functions of time, μ is a constant. Evidently, one of the solutions of (0.1)

$$p = q = \gamma_1 = \gamma_2 = 0, \qquad r = r_0, \ \gamma_3 = 1 \qquad (0.2)$$

corresponds to body rotation around an axis of symmetry coinciding with the direction to the center of attraction, at a constant angular velocity.

1. We obtain sufficient conditions for the stability of the motion (0.2) from the equation for the angle of nutation θ. It follows from the equations of motion (0.1)

$$p^2 + q^2 + a\gamma_3 - \mu(1-\delta)\gamma_3^2 - \int_0^t \gamma_3 da - \mu\int_0^t \gamma_3^2 d\delta = C_1 \qquad (1.1)$$

$$p\gamma_1 + q\gamma_2 + (r\delta + \nu)\gamma_3 - \int_0^t \gamma_3 d(r\delta + \nu) = C_2, \quad r = r_0 \tag{cont.}$$

where C_1, C_2, r_0 are constants. We obtain the first relationship in (1.1) from (0.1) by multiplying the first two equations by p and q, respectively, and adding, and the second by multiplying the first and second equations of (0.1) by γ_1 and γ_2, respectively, the fourth and fifth by p and q, respectively, and adding. We then integrate by parts in the integrals obtained.

Expressing $p, q, r, \gamma_1, \gamma_2, \gamma_3$ in terms of the Euler angles φ, ψ, θ we rewrite the relationships (1.1) as:

$$\psi'^2 \sin^2\theta + \theta'^2 + a\cos\theta - \mu(1-\delta)\cos^2\theta - \int_0^t \cos\theta da - \mu\int_0^t \cos^2\theta d\delta = C_1 \tag{1.2}$$

$$\psi' \sin^2\theta + (r_0\delta + \nu)\cos\theta - \int_0^t \cos\theta d(r_0\delta + \nu) = C_2$$

Eliminating the derivative ψ' from (1.2), with the notation $\cos\theta = u$, yields the equation

$$u'^2 = (1-u^2)\left[C_1 - au + \mu(1-\delta)u^2 + \int_0^t u da + \mu\int_0^t u^2 d\delta\right] -$$

$$-\left[C_2 - (r_0\delta + \nu)u + \int_0^t u d(r_0\delta + \nu)\right]^2 = f(u) \tag{1.3}$$

As is seen from (1.3), a change in the function u within the interval $(-1, 1)$, where $f(u) \geqslant 0$, corresponds to real motion of a body around a fixed point.

A change in the function u within the interval $(1-\varepsilon, 1)$, where ε is an arbitrarily small positive quantity ($\varepsilon \ll 1$), corresponds to the motion (0.2) of a body whose stability is investigated.

Chetaev [4] and Aminov [2] showed in investigations of the rotational motions of projectiles that the sufficient condition for u to be close to unity, i.e. the stability of the motion (0.2), is that the roots of the polynomial $\Phi(x) = -F(1-\varepsilon-x)$ are negative, where $F(u) \geqslant f(u)$. For a function $f(u)$ of the form (1.3) the polynomial $F(u) > f(u)$ is easily found if a, ν and δ are monotone functions of time.

For definiteness, let us set $a' \leqslant 0$, $\delta' \geqslant 0$, $\nu' \geqslant 0$. Then

$$\int_0^t u da \leqslant \int_0^t u_{\min} da \leqslant (1-\varepsilon)(a - a_0), \qquad \int_0^t u^2 d\delta \leqslant \int_0^t u_{\max}^2 d\delta = \delta - \delta_0$$

$$\int_0^t u d(r_0\delta + \nu) \geqslant \int_0^t u_{\min} d(r_0\delta + \nu) = (1-\varepsilon)(r_0\delta + \nu - r_0\delta_0 - \nu_0) \tag{1.4}$$

where $a_0 = a(0)$, $\delta_0 = \delta(0)$, $\nu_0 = \nu(0)$. Under conditions (1.4) the function $f(u)$ will evidently not exceed a fourth degree polynomial in u

$$F(u) = (1-u^2)[\mu(1-\delta)u^2 - au + C_1 + (1-\varepsilon)(a - a_0) + \mu(\delta - \delta_0)] - [C_2 - (r_0\delta + \nu)u + (1-\varepsilon)(r_0\delta + \nu - r_0\delta_0 - \nu_0)]^2 \geqslant f(u) \tag{1.5}$$

The polynomial $\Phi(x)$ whose roots must be investigated, becomes

$$\Phi(x) = -F(1-\varepsilon-x) = kx^4 + lx^3 + mx^2 + nx + b \tag{1.6}$$

where

$$k = \mu(1-\delta), \qquad l = a - 4\mu(1-\varepsilon)\ (1-\delta)$$

$$m = (r_0\delta + \nu)^2 + C_1 + 5\mu(1-\delta)\ (1-\varepsilon)^2 + \mu(\delta - \delta_0) -$$
$$- 2\varepsilon\mu(1-\varepsilon)\ (1-\delta) - 2(1-\varepsilon)a - (1-\varepsilon)\ a_0$$

$$n = 2(r_0\delta + \nu)[C_2 - (1-\varepsilon)\ (r_0\delta_0 + \nu_0)] + 2(1-\varepsilon)^2 a_0 - 2\mu\ (1-\varepsilon)\ (\delta - \delta_0) -$$
$$- 2(1-\varepsilon)C_1 - 2\mu(1-\varepsilon)^3(1-\delta) - 4\varepsilon\mu(1-\varepsilon)^2\ (1-\delta) - 2\varepsilon(1-\varepsilon)a$$

$$b = [C_2 - (1-\varepsilon)\ (r_0\delta_0 + \nu_0)]^2 - 2\varepsilon(1-\varepsilon)\ [\mu(1-\delta)\ (1-\varepsilon)^2 +$$
$$+ C_1 - (1-\varepsilon)a_0 + \mu(\delta - \delta_0)$$

According to the Hurwitz criterion, the inequalities

$$l > 0, \quad lm - kn > 0, \quad n(lm - kn) - l^2 b > 0$$

$$(lm - kn)\ (n^2 - bm) - bl^2 n > 0 \tag{1.7}$$

are the conditions for the roots of the polynomial (1.6) to be negative.

The inequalities (1.7) will be sufficient conditions for the stability of motion (0.2).

2. Sufficient conditions for the stability of motion (0.2) are obtained by using Liapunov functions. In perturbed motion we set

$$p = x_1, \; q = x_2, \; r = r_0 + x_3, \; \gamma_1 = y_1, \; \gamma_2 = y_2, \; \gamma_3 = 1 + y_3 \tag{2.1}$$

Then the perturbed motion equations for the variables x_1, y_1, x_2, y_2 have the form

$$x_1^{\cdot} = \alpha_1 x_2 + \alpha_2 y_2 + \{2\}, \qquad \alpha_1 = (1-\delta)\ r_0 - \nu$$

$$x_2^{\cdot} = -\alpha_1 x_1 - \alpha_2 y_1 + \{2\}, \qquad \alpha_2 = {}^1/_2 a - \mu\ (1-\delta) \tag{2.2}$$

$$y_1^{\cdot} = r_0 y_2 - x + \{2\}, \qquad y_2^{\cdot} = x_1 - r_0 y_1 + \{2\}$$

where $\{2\}$ are terms of second and higher order in the perturbations. We take as Liapunov function

$$V = \alpha x_1^2 + 2\beta x_1 y_1 + \gamma y_1^2 + \alpha x_2^2 + 2\beta x_2 y_2 + \gamma y_2^2 \qquad \left(\beta = -\frac{\alpha x_2 + \gamma}{r_0\delta - \nu}\right) \tag{2.3}$$

Here α, γ, β are continuous and bounded functions of time together with their derivatives.

Because of the perturbed motion equations (2.2) the derivative of the function (2.3) takes the form

$$V^{\cdot} = \alpha^{\cdot} x_1^2 + 2\beta^{\cdot} x_1 y_1 + \gamma^{\cdot} y_1^2 + \alpha^{\cdot} x_2^2 + 2\beta^{\cdot} x_2 y_2 + \gamma^{\cdot} y_2^2 + \{3\} \tag{2.4}$$

where $\{3\}$ are terms of third and higher order in the perturbations. Under the conditions

$$\alpha > 0, \quad \alpha^{\cdot} < 0, \quad \alpha\gamma - \beta^2 > 0, \qquad \alpha^{\cdot}\gamma^{\cdot} - \beta^{\cdot 2} > 0 \tag{2.5}$$

The function V is positive definite, and $V^{\cdot}$ is negative definite in the variables x_1, x_2, y_1, y_2.

According to the theorem proved in [5], the motion (0.2) is stable relative to part of the variables p, q, γ_1, γ_2 upon compliance with the inequalities (2.5). Then stability relative to the variables r and γ_3 follows from the third equation of (0.1) and the trivial integral $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$.

In particular, $\beta = -A/(r_0\delta + \nu)$, $\quad \gamma = A(1 - \alpha_2)$ for $\alpha = A$ and we obtain sufficient conditions for the stability of the motion (0.2) presented in [3]. In the case under consideration, a set of such conditions can be obtained. Thus in the case $\nu^{\cdot} < 0$, $\delta^{\cdot} < 0$, we can take

$$\alpha = r_0\delta + \nu, \quad \gamma = \mu(r_0\delta + \nu), \; -\beta = \mu\delta + {}^1/_2 a$$

Then conditions (2.5) have the form

$$\mu(r_0\delta + \nu)^2 - (\mu\delta + {}^1/_2 a)^2 > 0, \qquad \mu(r_0\delta^\cdot + \nu^\cdot)^2 - (\mu\delta^\cdot + {}^1/_2 a^\cdot)^2 > 0 \qquad (2.6)$$

The motion (0.2) of a body is hence stable.

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Translated by M. D. F.

KINEMATIC INTERPRETATION OF THE MOTION OF A BODY IN THE HESS' SOLUTION

PMM Vol. 34, №3, 1970, pp. 567-570

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(Received December 31, 1968)

Kinematic interpretation of the motion of a body is based on kinematic equations put forward by Kharlamov [1]. The moving angular velocity hodograph was considered in our earlier paper [2] in which we classified all the characteristic forms of the moving hodograph. In the present paper we shall consider the stationary hodograph in all these cases and give a geometric picture of the motion of a body.

1. The motion of the body can be described as slipless rolling of the moving axoid of angular velocity vector on the stationary axoid. The moving hodograph in the Hess' solution was already fully studied in [2], and we shall make use of the results of this study and take the same notation.

The moving hodograph lies in the plane $\omega_1 = {}^1/_2\, c\omega_2$; its projection on the plane $\omega_1 = 0$ is the curve s the equations of which in polar coordinates ρ and $\varphi(\omega_2 = \rho\cos\varphi$, $\omega_3 = \rho\sin\varphi)$ have the form

$$\rho\rho^\cdot = \sqrt{f(\rho)}, \qquad \rho^2\varphi^\cdot = -\rho^3\cos\varphi + k$$

$$\left(f(\rho) = \rho^2\left[1 - \left(\frac{\rho^2}{c} - h\right)^2\right] - k^2\right) \qquad (1.1)$$

the dot superscript denotes differentiation with respect to the dimensionless time τ.